

## **Perturbations of a Universe Filled with Dust and Radiation**

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A first-order perturbation approach to  $k = 0$  Friedmann cosmologies filled with dust and radiation is developed. Adopting the coordinate gauge comoving with the perturbed matter, and neglecting the vorticity of the radiation, a pair of coupled equations is obtained for the trace  $h$  of the metric perturbations and for the velocity potential  $v$ . A power series solution with upward cutoff exists such that the leading terms for large values of the dimensionless time  $\xi$  agree with the relatively growing terms of the dust solution of Sachs and Wolfe.

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### **1. INTRODUCTION**

The classic prediction for the temperature fluctuations of the cosmic background radiation by Sachs and Wolfe (1967) overestimates the experimental value (Mather *et al.*, 1992) by at least two orders of magnitude. Worse than that, this prediction has been obtained by neglecting the temperature fluctuations on the surface of the last scattering, and including only the gravitational perturbations along the subsequent path of the photon. Obviously, if the initial fluctuations are random, their inclusion can only increase the effect. While these computations can be criticized on the basis that the results are not invariant with respect to the choice of the initial hypersurface where the photons originate (Stoeger, 1991), this is not likely to be the way of improving the results. Meanwhile, the discovery of large-scale structures (such as voids or walls) certainly did not bring us closer to the solution of this cosmic puzzle. It is hard to take seriously the suggestion (Mather *et al.*, 1992) that the observed fluctuations are primordial (as opposed to propagation effects), unless we are able to reduce the magnitude of the Sachs–Wolfe

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estimate accordingly. The way the surface of photon emission is defined is just one detail of the gauge choice. As we seek diminution of the effect, we may as well stick to the comoving gauge. For gauge-invariant treatments, see Magueijo (1993), Russ *et al.* (1993), Kodama and Sasaki (1987), Dunsby (1991), and Mukhanov *et al.* (1992).

Sachs and Wolfe obtained, in a closed form, the first-order perturbations of a Friedmann universe with a flat 3-space and filled either with dust or radiation. They assumed that the domain of the universe in which the photon travels is matter dominated, and computed the temperature fluctuations in the dust-filled universe. Their assumption is justifiable because the decoupling occurs near the time of equal matter and radiation densities. [Some authors estimate that the two phenomena occur simultaneously (Weld, 1984), while others (Lyth and Stewart, 1990) take that the equal-density epoch precedes the surface of last scattering.] According to the unperturbed models, the radiation density  $\rho_r$  dies out faster than the matter density  $\rho_m$ . However, due to the instabilities, the dust dominance may not hold everywhere in the perturbed models.

Here we consider a refinement of the Sachs–Wolfe scheme by computing the perturbations of the  $k = 0$  Friedmann universe in the presence of *both* dust and radiation. Instabilities are known to exist in two-fluid cosmologies (Mukhanov *et al.*, 1992). Assuming, for instance, that the observed large structures are lately emerging manifestations of the instabilities, one might be able to reduce the magnitude of the metric fluctuations affecting the photon orbits. The energy-momentum tensor is a sum of those of the two media,

$$T_b^a = T_{mb}^a + T_{rb}^a \quad (1)$$

The contribution of the dust has the form

$$T_{mb}^a = -\rho_m u^a u_b \quad (2)$$

For the radiation,

$$T_{rb}^a = -\frac{4}{3} \rho_r u^a u_b + \frac{1}{3} \rho_r \delta_b^a \quad (3)$$

The four-velocities are normalized,  $u^a u_a = 1$ . We adopt the conformal form of the metric

$$g_{ab} = a^2(\eta)(\eta_{ab} + h_{ab}) \quad (4)$$

where the unperturbed ( $h_{ab} = 0$ ) metric satisfies (Misner *et al.*, 1973)

$$3 \frac{1}{a^2} \left( \frac{da}{d\eta} \right)^2 - \frac{\rho_{m0} a_0^3}{a} - \frac{\rho_{r0} a_0^4}{a^2} = 0 \quad (5)$$

such that the density  $\rho_i$ , where  $i$  stands either for  $m$  (matter) or  $r$  (radiation),

equals  $\rho_{m0}$  or  $\rho_{r0}$  at some prescribed conformal time  $\eta = \eta_0$ . The solution has the form

$$a = \frac{1}{4} \lambda \eta^2 - \mu \tag{6}$$

where the constants are defined by

$$\lambda = \frac{1}{3} \rho_{m0} a_0^3, \quad \mu = \frac{\rho_{r0}}{\rho_{m0}} a_0 \tag{7}$$

## 2. THE PERTURBED MODEL

In the perturbed universe,  $h_{ab} \neq 0$ , the densities of the components can be written to first order

$$\rho_i^{(1)} = \rho_i + \delta\rho_i \tag{8}$$

Here  $\rho_r$  and  $\rho_m$  are the unperturbed densities

$$\rho_m(\eta) = \rho_{m0} \frac{a_0^3}{a^3}, \quad \rho_r(\eta) = \rho_{r0} \frac{a_0^4}{a^4} \tag{9}$$

and  $\delta\rho_i$  are the first-order density perturbations. The indices of the perturbed quantities are lowered and raised by the Minkowski metric  $\eta_{ab} = \eta^{ab}$ .

It can be proven (J. Ehlers, personal communication, 1990) that a homogeneous universe cannot develop perturbations with comoving dust and radiation. In fact, the dipole effect of the cosmic background radiation provides an experimental value for the local relative velocity of the order of 100 km/sec. Thus we have good reason to assume that  $\delta u_m^i \neq \delta u_r^i$ . We then choose coordinates comoving with the matter:

$$u_m^a = u_0^a \tag{10}$$

$$u_r^a = u_0^a + \delta u_r^a \tag{11}$$

where the coincident unperturbed velocities are

$$u_0^a = \frac{1}{a} \delta_0^a \tag{12}$$

The normalization conditions imply that  $h_{00} = 0$  and  $\delta u_r^0 = \delta u_{r0} = 0$ .

After the decoupling, we may assume that the conservation laws apply separately both to the matter and the radiation components:  $T_{mb;a}^a = 0$  and  $T_{rb;a}^a = 0$ . Thus coupling occurs only via the universal gravitational interaction. We get two energy conservation equations from the timelike components and two momentum conservations laws from the spacelike components.

## 2.1. Energy Conservation

The first-order perturbations of the *dust* satisfy the equation

$$\left( \frac{\delta\rho_m}{\rho_m} + \frac{1}{2} h \right)' = 0 \quad (13)$$

where  $h = h_\alpha^\alpha$  is the trace of the spacelike perturbation of the metric and a prime denotes partial derivative with respect to the conformal time  $\eta$ . The solution is

$$\delta\rho_m = \rho_m \left( E(x^\beta) - \frac{1}{2} h \right) \quad (14)$$

where the integration function  $E(x^\beta)$  depends only on the space coordinates  $x^\alpha$  ( $\alpha = 1, 2, \text{ or } 3$ ).

For the *radiation*, the energy conservation law has the form

$$(\rho_r^{3/4} \sqrt{g} u_r^\alpha)_{,\alpha} = 0 \quad (15)$$

or

$$\left( \frac{3}{4} \frac{\delta\rho_r}{\rho_r} + \frac{1}{2} h \right)' + a(\delta u_r^\alpha)_{,\alpha} = 0 \quad (16)$$

## 2.2. Momentum Conservation

The conservation law for the *dust* has the form

$$(ah_{0\alpha})' = 0 \quad (17)$$

with the solution

$$ah_{\alpha 0} = F_\alpha(x^\beta) \quad (18)$$

The remaining coordinate freedom makes it possible to arrange (White, 1973)

$$h_{,\alpha}^{0\alpha} = 0 \quad (19)$$

For *radiation*, the first-order momentum conservation law reads

$$(\rho_r^{1/4} \delta u_{r\alpha})' = \frac{1}{4} a \rho_r^{-3/4} \delta \rho_{r,\alpha} \quad (20)$$

The perturbed Einstein tensor will be written

$$G^a_b = {}_0G^a_b + \delta G^a_b \quad (21)$$

Here  ${}_0G^a_b$  is the unperturbed and  $\delta G^a_b$  the first-order part. The Einstein equations for the first-order quantities are

$$\delta G^0_0 = -(\delta\rho_r + \delta\rho_m) \tag{22}$$

$$\delta G^\alpha_0 = -\frac{4}{3} a\rho_r\delta u_r^\alpha \tag{23}$$

$$\delta G^\alpha_\beta = \frac{1}{3} \delta^\alpha_\beta\delta\rho_r \tag{24}$$

Substitution of the Sachs and Wolfe (1967) expressions for  $\delta G^a_b$  and separating the trace-free part of the metric perturbation

$$S_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{3} \eta_{\alpha\beta}h \tag{25}$$

yields

$$S^{\mu\nu}{}_{,\mu\nu} + \frac{2}{3} \Delta h - 2\frac{a'}{a} h' = -2a^2(\delta\rho_r + \delta\rho_m) \tag{26}$$

$$S^{\alpha\mu}{}_{,\mu}{}' - \frac{2}{3} h^{\alpha\prime} + \Delta h^{0\alpha} - 4\left(2\frac{a'^2}{a^2} - \frac{a''}{a}\right)h^{0\alpha} = -\frac{4}{3} a\rho_r\delta u_r^\alpha \tag{27}$$

$$2h'' + 4\frac{a'}{a} h' - S^{\mu\nu}{}_{,\mu\nu} - \frac{2}{3} \Delta h = -2a^2\delta\rho_r \tag{28}$$

$$\begin{aligned} S^\alpha{}_\beta{}'' + 2\frac{a'}{a} S^\alpha{}_\beta{}' - \Delta S^\alpha{}_\beta &= S^{\alpha\mu}{}_{,\beta\mu} + S_{\beta\mu,\alpha\mu} - \frac{2}{3} \delta^\alpha{}_\beta S^{\mu\nu}{}_{,\mu\nu} \\ &+ h^{\alpha 0}{}_{,\beta}{}' + h_{\beta 0,\alpha'} + 2\frac{a'}{a} (h^{\alpha 0}{}_{,\beta} + h_{\beta 0,\alpha}) \\ &- \frac{1}{3} h_{,\beta}{}^\alpha{}_\alpha - \frac{1}{9} \delta^\alpha{}_\beta \Delta h \end{aligned} \tag{29}$$

Taking the sum of (26) and (28), we get the simple relation

$$h'' + \frac{a'}{a} h' = -a^2(2\delta\rho_r + \delta\rho_m) \tag{30}$$

### 3. THE VELOCITY POTENTIAL

For economy of writing, we introduce the scaled radiation velocity perturbation

$$v_\alpha(x_\alpha, \eta) = \rho_r^{1/4}\delta u_{r\alpha} \tag{31}$$

(Note that the component  $v_0 = 0$ .) From the  $\eta$  derivative of (16) and eliminating the term  $a^2(\delta u_r^\beta)$  by use of the radiation momentum conservation (20), we get the relation

$$\left[ \rho_r^{1/4} a \left( \frac{3}{4} \frac{\delta \rho_r}{\rho_r} + \frac{1}{2} h \right) \right]' + \frac{1}{4} a \rho_r^{-3/4} (\delta \rho_r)_{, \alpha} = 0 \tag{32}$$

Taking the time derivative and the gradient, and using again (20) to get rid of the divergence term, the left-hand side is still a total time derivative. Integration and comparison with (32) leads to the uncoupled equation

$$v_{\beta, \alpha}{}^\alpha - v^\alpha{}_{, \alpha \beta} = Q_\beta(x) \tag{33}$$

where the integration function  $Q_\beta(x)$  depends only on space coordinates. We decompose the three-vector  $v_\alpha$

$$v_\alpha = \omega_\alpha + v_{, \alpha} \tag{34}$$

where the *vorticity*  $\omega_\alpha$  is divergenceless

$$\omega_{\alpha, \alpha}{}^\alpha = 0 \tag{35}$$

and  $v$  is the *velocity potential*. The terms containing the velocity potential  $v$  cancel in equation (33). Thus we get the simple relation for the vorticity

$$\Delta \omega_\beta = -Q_\beta(x) \tag{36}$$

Following a suggestion of Liddle and Lyth (1993), we shall henceforth take the trivial solution  $\omega_\alpha = 0$  and  $Q_\beta = 0$  such that we have

$$v_\alpha = v_{, \alpha} \tag{37}$$

Using the velocity potential in equation (20), we get

$$v'_{, \alpha} = \frac{1}{4} a \rho_r^{-3/4} \delta \rho_{r, \alpha} \tag{38}$$

This can be integrated. Since only the gradient of the velocity potential has physical meaning, the integration function  $U(\eta)$  may be chosen to vanish. Thus

$$v' = \frac{1}{4} a \rho_r^{-3/4} \delta \rho_r \tag{39}$$

Hence, equation (16) can be rewritten

$$3v'' - \Delta v + \frac{1}{2} \rho_{r0}^{1/4} a_0 h' = 0 \tag{40}$$

Introducing the velocity potential (39) in (30), we get

$$h'' + \frac{a'}{a} h' + \rho_{m0} \frac{a_0^3}{a} \left( E(x) - \frac{1}{2} h \right) + 8a\rho_r^{3/4} v' = 0 \tag{41}$$

We may get rid of the inhomogeneous term  $E(x)$  by introducing the function

$$f = h - 2E(x) \tag{42}$$

The explicit form of the scale factor  $a$  can be made simpler by use of the dimensionless time variable

$$\xi = \frac{1}{2} \left( \frac{\lambda}{\mu} \right)^{1/2} \eta \tag{43}$$

Then we have

$$a = \mu(\xi^2 - 1), \quad \mu = \frac{\rho_{r0}}{\rho_{m0}} a_0 \tag{44}$$

such that the Big Bang occurs at  $\xi = 1$ . Equations (40) and (41) take the form

$$\dot{v} - L\Delta v + \frac{16}{K} \dot{f} = 0 \tag{45}$$

$$(\xi^2 - 1)\ddot{f} + 2\xi\dot{f} - 6f + K \frac{1}{\xi^2 - 1} \dot{v} = 0 \tag{46}$$

where an overdot denotes derivative with respect to the dimensionless time  $\xi$  and the constants  $K$  and  $L$  are defined by

$$K = 16\sqrt{3}\rho_{m0}\rho_{r0}^{-3/4} \tag{47}$$

$$L = 4 \frac{\rho_{r0}}{\rho_{m0}^2} \frac{1}{a_0^2} \tag{48}$$

These coupled linear equations for  $f$  and  $v$  have an elaborate structure, even in the asymptotic regime. In the special case of the velocity potential being stationary,  $\dot{v} = 0$ , (46) becomes the  $n = 2$  Legendre equation for  $f$ . In the generic case, however, the expansion in Legendre series is cumbersome.

### 4. RECURSION RELATIONS

The power series expansion in the time variable  $\xi$ ,

$$f(x, \xi) = \sum_{n=-\infty}^{\infty} a_n(x)\xi^n \tag{49}$$

$$v(x, \xi) = \sum_{n=-\infty}^{\infty} b_n(x)\xi^n \tag{50}$$

with equations (45) and (46), yields the recursion relations for the coefficients:

$$(n + 1)(n + 2)b_{n+2} - L\Delta b_n + \frac{16}{K}(n + 1)a_{n+1} = 0 \tag{51}$$

$$(n + 1)(n - 4)a_{n-2} - (2n^2 - 6)a_n + (n + 1)(n + 2)a_{n+2} + K(n + 1)b_{n+1} = 0 \tag{52}$$

Eliminating  $a_n$ , we get

$$(n + 1)(n - 1)(n - 4)b_{n-1} - (n + 1)(2n^2 + 10)b_{n+1} + (n + 1)(n + 2)(n + 3)b_{n+3} - \frac{L}{n - 2}(n + 1)(n - 4)\Delta b_{n-3} + L \frac{2n^2 - 6}{n} \Delta b_{n-1} - L(n + 1)\Delta b_{n+1} = 0 \tag{53}$$

The convergence of the series expansion deserves further investigation. The terms with negative exponents will decay with time, and it is expected that their contribution to the temperature fluctuations becomes negligible. Hence, to evade fast-growing terms, it appears worthwhile to seek solutions with a maximal value of the exponent  $n$ .

It is clear from the structure of equations (51) and (52) that a cutoff of the power series can occur only simultaneously in  $a_n$  and  $b_n$ . Further necessary conditions for cutoff may be obtained from (53). The highest possible nonvanishing coefficient is  $b_1$ . From (51) it then follows that  $a_2$  is the highest coefficient of  $f$ . It is impossible to have a cutoff simultaneously both upward and downward. Our choice of the velocity potential implies that any coefficient  $b_n$  satisfying  $\Delta b_n = 0$  must vanish. Thus the leading terms of the solution of equations (51) and (51) are

$$h(x, \xi) = a_2(x)\xi^2 + \left(2E - \frac{1}{3} a_2(x)\right) + a_{-2}(x) \frac{1}{\xi^2} + a_{-3}(x) \frac{1}{\xi^3} + \frac{1}{3} a_{-2}(x) \frac{1}{\xi^4} + \dots \tag{54}$$



$$v(x, \xi) = b_1(x)\xi + b_{-3}(x)\frac{1}{\xi^3} + b_{-4}(x)\frac{1}{\xi^4} + \dots \tag{55}$$

where the coefficients  $a_n$  and  $b_n$  satisfy the relations

$$\begin{aligned} a_0(x) &= -\frac{1}{3} a_2, & \Delta b_1(x) &= \frac{32}{KL} a_2(x) \\ a_{-2}(x) &= \frac{K}{2} b_1(x), & \Delta b_{-3}(x) &= \frac{8}{L} b_1 \\ a_{-4}(x) &= \frac{a_{-2}(x)}{3}, & \Delta b_{-4}(x) &= \frac{48}{KL} a_{-3}(x) \end{aligned}$$

The leading terms in the density contrasts are

$$\frac{\delta\rho_r}{\rho_r} = 4\rho_{r0}^{-1/4} a_0^{-1} \left( b_1(x) - 3b_{-3}(x)\frac{1}{\xi^4} - 4b_{-4}(x)\frac{1}{\xi^5} + \dots \right) \tag{56}$$

$$\frac{\delta\rho_m}{\rho_m} = -\frac{1}{2} \left( a_2(x)\xi^2 + a_0(x) + a_{-2}(x)\frac{1}{\xi^2} + a_{-3}(x)\frac{1}{\xi^3} + \dots \right) \tag{57}$$

Thus we establish the comforting result that for large values of  $\xi$ , the leading power in time of the metric function  $h$  coincides with the power of the relatively growing mode of the pure dust solution (Sachs and Wolfe, 1967). For a more detailed account of our approach, see Komárik (1994).

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